Harmonic Mappings Related to the Bounded Boundary Rotation

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Abstract—The aim of this paper is to give investigation of the class of harmonic functions related to the bounded boundary rotation. The class of bounded boundary rotation is generalized to the convex function. For this aim we use subordination techniques to obtain known as well as new results related to the Libera-type problem [1].

Keywords—Harmonic mapping, bounded rotation, boundary rotation, starlike function, convex function.

I. Introduction

Let $\Omega$ be the class of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0, |\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ regular in $\mathbb{D}$, such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}$$

(1.1)

for some functions $\phi(z) \in \Omega$, and every $z \in \mathbb{D}$.

$\mathcal{P}_k$ denotes the class of functions $p(0) = 1$ and analytic in $\mathbb{D}$ with the representation

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

(1.2)

where $\mu(t)$ is defined by

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$  (1.3)

Clearly $p_2 = p$. From the (1.2), one can easily find that $p(z) \in \mathcal{P}_k$ can also be written as

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$  (1.4)

where $p_1(z), p_2(z) \in \mathcal{P}$.

Moreover, let $\mathcal{A}$ be the class of all analytic functions of the form $s(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ which are regular in $\mathbb{D}$. Let $\mathcal{C}$ denote the family of functions $s(z) \in \mathcal{A}$ such that $s(z)$ is in $\mathcal{C}$ if and only if

$$1 + z \frac{s''(z)}{s(z)} = p(z)$$

(1.5)

for some $p(z) \in \mathcal{P}$, and all $z \in \mathbb{D}$. The class $\mathcal{C}$ is called the class of convex functions, and let $s(z)$ be an element of $\mathcal{A}$ if the equality

$$\frac{s'(z)}{s(z)} = p(z)$$

is satisfied for some $p(z) \in \mathcal{P}$ and every $z \in \mathbb{D}$, then $s(z)$ is called starlike functions, the class of functions is denoted by $\mathcal{S}^*$. Let $s(z)$ be an element of $\mathcal{A}$ and which maps $\mathbb{D}$ conformally onto an image domain of boundary rotation at most $k\pi$, the class of such functions is denoted by $V(k)$. The concept of functions of bounded boundary rotation original from Loewner [3] in 1917 but he did not use the present terminology. Paatero [2], who systematically developed their properties and made an exhaustive study of the class $V(k)$. Paatero [2] has shown that $s(z) \in V(k)$ if and only if

$$s'(z) = E \exp\left[-\int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t)\right],$$

(1.6)

where $\mu(t)$ is given in (1.3). For a fixed $k \geq 2$ it can also be expressed as

$$\int_0^{2\pi} \left| \frac{R(zs(z))'}{s(z)} \right| d\theta \leq k\pi, \quad z = re^{i\theta}. \quad (1.7)$$

Clearly, if $k_1 < k_2$ then $V(k_1) \subset V(k_2)$, that is the class $V(k)$ obviously expand as $k$ increases. $V(2)$ is simply the class of $\mathcal{C}$ convex univalent functions and Paatero [2] showed that $V(4) \subset \mathcal{S}$, where $\mathcal{S}$ is the class of normalized univalent functions. Later Pinchuk [4] proved that functions in $V(k)$ are close-to-convex in $\mathbb{D}$ if $2 \leq k \leq 4$ and hence univalent. A function $s(z)$ analytic in $\mathbb{D}$ is said to be close-to-convex, if there exists a function $\ell(z) \in \mathcal{C}$ such that

$$\Re\left(\frac{s'(z)}{\ell(z)}\right) > 0$$

(1.8)

for all $z \in \mathbb{D}$. Let $s(z)$ be an element of $\mathcal{A}$ if $s(z)$ having the representation

$$s(z) = z \exp\left[-\int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t)\right],$$

(1.9)

where $\mu(t)$ is given in (1.3), then $s(z)$ is called bounded radius rotation. The class of these functions is denoted by $R(k)$. Pinchuk [4] also showed that Alexander type relation between the classes $V(k)$ and $R(k)$ is

$$s(z) \in V(k) \quad \text{if and only if} \quad (zs'(z)) \in R(k). \quad (1.10)$$

Pinchuk [4] has shown that the classes $V(k)$ and $R(k)$ can be defined by using class $\mathcal{P}_k$ as given below

$$s(z) \in V(k) \quad \text{if and only if} \quad (zs'(z))' \in \mathcal{P}_k, \quad (1.11)$$

$$s(z) \in R(k) \quad \text{if and only if} \quad \frac{zs'(z)}{s(z)} \in \mathcal{P}_k. \quad (1.12)$$

Let $s_1(z)$ and $s_2(z)$ be elements of $\mathcal{A}$. If there exist a function $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$, then
we say that $s_1(z)$ is subordinate to $s_2(z)$, and we write $s_1(z) \prec s_2(z)$. If $s_2(z)$ is univalent in $\mathbb{D}$, then $s_1(z) \prec s_2(z)$ if and only if $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$, $s_1(0) = s_2(0)$ implies $s_1(\mathbb{D}_r) \subset s_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r, 0 < r < 1\}$.

(Subordination principle and Lindelöf principle [5]).

Finally, a planar harmonic mapping $f$ in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function which maps $\mathbb{D}$ onto the same planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping $f$ has a canonical decomposition $f = h(z) + g(z)$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series

\[
h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad a_n, b_n \in \mathbb{C}
\]

We call $h(z)$ an analytic part of $f$ and $g(z)$ is co-analytic part of $f$. An elegant and complete treatment of harmonic mapping is given in Duren’s monograph [6]. Lewy [6] proved that the harmonic mapping $f$ is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |(h'(z))^2 - (g'(z))^2|$ is different from zero in 1936. In view of this result locally univalent harmonic mapping in the open unit disc $\mathbb{D}$ are either sense-preserving $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ or sense-reversing $|h'(z)| < |g'(z)|$ in $\mathbb{D}$. In this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h(z) + g(z)$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in $\mathbb{D}$ and the second dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the class of all sense-preserving harmonic mappings in the open unit disc $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by $S_H$. Thus $S_H$ contains standard class $S$ of univalent functions. The family of all mappings $f \in S_H$ with the additional property $g'(0) = 0$, i.e., $b_1 = 0$ is denoted by $S_{H1}$. Hence it is clear that $S \subset S_{H1} \subset S_H$.

We will investigate the following class

\[
S_{HV}(k) = \left\{ f = h(z) + g(z) \mid \frac{1}{h'(z)} g'(z) \in P_K, \quad h(z) \in V(K) \right\}.
\]

For our proofs we need the following lemma and theorems.

**Lemma I.1.** ([7]) Let $\phi(z)$ be a regular in the open unit disc $\mathbb{D}$ with $\phi(0) = 0$, then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_0$ has $z_0 \phi'(z_0) = m \phi(z_0)$, $m \geq 1$.

**Theorem I.2.** ([8]) Let $s(z)$ be an element of $V(K)$ then

\[
\frac{(1 - r)^{\frac{k - 1}{k}}}{(1 + r)^{\frac{k + 1}{k}}} \leq |s'(z)| \leq \frac{(1 + r)^{\frac{k - 1}{k}}}{(1 - r)^{\frac{k + 1}{k}}}, \quad \text{(I.13)}
\]

\[
M_2(a, b, c, r) \leq |s(z)| \leq M_1(a, b, c, r), \quad \text{(I.14)}
\]

where

\[
M_1(a, b, c, r) = \frac{2^{b-1}}{a} [G(a, b, c, -1) - r^{-1} G(a, b, c, r^{-1})],
\]

\[
M_2(a, b, c, r) = \frac{2^{b-1}}{a} [G(a, b, c, -1) - r^{-1} G(a, b, c, r^{-1})],
\]

\[
a = \frac{k}{2}, \quad b = 0, \quad c = \frac{k}{2} + 1, \quad r_1 = \frac{1 - r}{1 + r},
\]

\[
G(a, b, c, r) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^r u^{a-1} (1 - u)^{c-a-1} (1 - zu)^{-b} du.
\]

**Theorem I.3.** ([9]) Let $k \geq 2$ and $s(z) \in V(K)$ such that $s(z) \neq 0$ in $\mathbb{D}$. Then

\[
G(z) = \left( \int_0^z s(t) \frac{dt}{t} \right) \in V(K).
\]

**Theorem I.4.** ([10]) Let $s(z) \in V(K)$, $2 \leq k < \infty$. Let $x \in \mathbb{D}$ and

\[
F(z) = s \left( \frac{z + x}{1 + xt} \right) - s(x).
\]

Then $F(z) \in V(K)$ and

\[
|z s''(z) - 2 r^2| \leq \frac{kr}{1 - r^2}.
\]

**II. Conclusion**

**Lemma II.1.** Let $p(z) \in P_k$ then

\[
|p(z) - \frac{1 + r^2}{1 - r^2}| \leq \frac{kr}{1 - r^2}.
\]

**Proof:** Using Theorem I.4, then we write

\[
|z \frac{s''(z)}{s'(z)} - \frac{2 r^2}{1 - r^2}| \leq \frac{kr}{1 - r^2}
\]

after the simple calculations we get

\[
\left| \left(1 + \frac{2 r^2}{1 - r^2}\right) - (1 - r^2) \right| \leq \frac{kr}{1 - r^2}.
\]

In this step by using the definition of the class $V(K)$ we can write

\[
|p(z) - \frac{1 + r^2}{1 - r^2}| \leq \frac{kr}{1 - r^2}.
\]

**Theorem II.2.** Let $f = h(z) + g(z)$ be an element of $S_{HV}(k)$, then $\frac{g(z)}{h'(z)}$ is in $V(K)$.

**Proof:** Using the subordination principle and definition of the class $S_{HV}(k)$, then we write $w(\mathbb{D}_r)$ as

\[
\left\{ \begin{array}{l}
\left| b_1 \left( \frac{1}{2} + \frac{k}{2} \right) \frac{1 + \phi_1(z)}{\phi_1(z) - \left( \frac{1}{2} + \frac{k}{2} \right) 1 + \phi_2(z)} \right| - \frac{1 + r^2}{1 - r^2} \\
\leq \frac{kr}{1 - r^2}, \quad k > 2
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\left| b_1 \left( \frac{1}{2} + \frac{k}{2} \right) \frac{1 + \phi_1(z)}{\phi_1(z) - \left( \frac{1}{2} + \frac{k}{2} \right) 1 + \phi_2(z)} \right| \leq \frac{2r}{1 - r^2}, \quad k = 2
\end{array} \right.
\]

where $\phi_1, \phi_2 \in \Omega, 0 < r < 1$. Now we define the function $p_k(z)$ by

friend
where \( \phi(z) = \frac{k}{4} + \frac{1}{2} \frac{1 + \phi_1(z)}{1 - \phi_1(z)} - \frac{k}{4} - \frac{1}{2} \frac{1 + \phi_2(z)}{1 - \phi_2(z)} \)

then \( p_k(z) \) is analytic and \( p_k(0) = 1 \), i.e., \( \phi_1(0) = 0, \phi_2(0) = 0 \)

\[
g(z) = \frac{1}{h(z)} = b_1 \left( \frac{k}{4} + \frac{1}{2} \frac{1 + \phi_1(z)}{1 - \phi_1(z)} - \frac{k}{4} - \frac{1}{2} \frac{1 + \phi_2(z)}{1 - \phi_2(z)} \right)
\]

In this step, if we use Theorem I.2 and Lemma II.1, then (II.2) can be written in the following manner

\[
g'(z) = \frac{\partial g}{\partial h} = \frac{1}{h(z)} \left( \frac{1}{h'z} \right) = \frac{b_1}{h'(z)} \frac{1}{1 + \phi_1(z)} \frac{1}{2} \frac{1 + \phi_2(z)}{1 - \phi_2(z)} - \frac{1}{2} \frac{1 + \phi_1(z)}{1 - \phi_1(z)} \frac{1}{1 - \phi_2(z)} \frac{1}{1 + \phi_2(z)}
\]

But this contradicts to (II.2) because \( \phi(z_0) = 1, m \geq 1 \). So our assumption \( \phi(z_0) = 1 \) is wrong, i.e., \( \phi(z) < 1 \) for all \( z \in D \), therefore we have \( g'(z) < b_1 p_k(z) \).

**Corollary II.3.** If \( f = h(z) + \frac{g}{h'}(z) \in S_{H\nu}(k) \), then

\[
M_2(a, b, c, r) = \frac{1 - kr + r^2}{1 - r^2} \leq |g(z)| \leq M_1(a, b, c, r) = \frac{1 + kr + r^2}{1 - r^2},
\]

\[
\frac{(1 - kr + r^2)(1 - r)^{\frac{1}{2} - 2}}{(1 + r)^{\frac{1}{2} + 2}} \leq |g'(z)| \leq \frac{(1 + kr + r^2)(1 + r)^{\frac{1}{2} - 2}}{(1 - r)^{\frac{1}{2} + 2}}
\]

where \( M_1(a, b, c, r) \) and \( M_2(a, b, c, r) \) are given Theorem I.2.

**Proof:** Using the definition \( S_{H\nu}(k) \) we can write

\[
|h(z)| \frac{1 - kr + r^2}{1 - r^2} \leq |g(z)| \leq \frac{1 + kr + r^2}{1 - r^2} |h(z)|,
\]

\[
|h'(z)| \frac{1 - kr + r^2}{1 - r^2} \leq |g'(z)| \leq \frac{1 + kr + r^2}{1 - r^2} |h'(z)|.
\]

In this step if we use Theorem I.2 then we obtain desired result.

**Corollary II.4.** Let \( f = h(z) + \frac{g}{h'}(z) \) be an element of \( S_{H\nu}(k) \), then

\[
\frac{(1 - kr + r^2)(1 - r)^{\frac{1}{2} - 2}}{(1 + r)^{\frac{1}{2} + 2}} \leq J_f
\]

\[
\frac{(1 + kr + r^2)(1 + r)^{\frac{1}{2} - 2}}{(1 - r)^{\frac{1}{2} + 2}}
\]

**Proof:** Since

\[
J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2 \left( 1 - \left| \frac{g'(z)}{h'(z)} \right|^2 \right)
\]

and in this step we use Corollary II.3, then we obtain desired result.

**Corollary II.5.** Let \( f = h(z) + \frac{g}{h'}(z) \in S_{H\nu}(k) \) then

\[
|f| \leq \int_0^1 (1 + r)^{\frac{1}{2} - 2} k - 1 kr + 2r^2 dr.
\]

**Proof:** Using the inequality

\[
(|h'(z)| - |g'(z)|) |dz| \leq |df| \leq (|h'(z)| + |g'(z)|) |dz|
\]

we get

\[
|h'(z)| \left( 1 - \left| \frac{g'(z)}{h'(z)} \right| \right) |dz| \leq |df| \leq |h'(z)| \left( 1 + \left| \frac{g'(z)}{h'(z)} \right| \right) |dz|
\]

after simple calculations we can get the result easily.

**REFERENCES**